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Scale dependent partitioning of one-dimensional aperiodic set diffraction

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Abstract. We give a multiresolution partition of pure point parts of diffraction patterns of one-dimensional aperiodic sets. When an aperiodic set is related to the Golden Ratio, denoted by τ, it is well known that the pure point part of its diffractive measure is supported by the extension ring of τ, denoted by $\mathbb{Z}[\tau]$. The partition we give is based on the formalism of the so called τ-integers, denoted by $\mathbb{Z}_\tau$. The set of τ-integers is a selfsimilar set obeying $\mathbb{Z}_\tau/\tau^{j-1} \subset \mathbb{Z}_\tau/\tau^j \subset \mathbb{Z}_\tau/\tau^{j+1} \subset \mathbb{Z}[\tau]$, $j \in \mathbb{Z}$. The pure point spectrum is then partitioned with respect to this "Russian doll" like sequence of subsets $\mathbb{Z}_\tau/\tau^j$. Thus we deduce the partition of the pure point part of the diffractive measure of aperiodic sets.

PACS. 61.44.Br Quasicrystals – 61.10.Dp Theories of diffraction and scattering

1 Introduction

Quasicrystals key feature is long-range order of non-translational type [1]. Diffraction patterns of generic structures display a pure point part, a singular continuous part, and an absolutely continuous part. The existence of a pure point part is an indication of order. Indeed, in the case of quasicrystals, Bragg peaks quasiperiodically span the reciprocal space, and obey some scaling law. The existence of an absolute continuous part is an indication of disorder. The singular continuous part, although it has been known since Lebesgue decomposition theorem of a measure, has not been encountered in Material Science until the discovery of quasicrystals, and is an indication of some intermediate state between quasiperiodicity and randomness, see [2] and [3,4].

In this article we focus on the pure point part of diffraction patterns of aperiodic sets which are generically denoted by Λ. Moreover we suppose that the sets Λ are generated by some Cut and Project scheme associated to the Golden Ratio, denoted here by τ. The pure point part of the diffraction pattern of a Λ is then the following weighted Dirac measure [5]

$$I(k) = \sum_{k\in\alpha\mathbb{Z}[\tau]} |c_k|^2 \delta_k\,,$$

where c_k is the Fourier coefficient of Λ for the wavelength k, $\alpha \in \mathbb{R}$ is some factor, and $\mathbb{Z}[\tau] = \{m + n\tau \mid m,n \in \mathbb{Z}\}$ is the extension ring of the Golden Ratio. The set $\mathbb{Z}[\tau]$ is the support of $I(k)$.

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There is a discrete subset of $\mathbb{Z}[\tau]$ which plays a particular importance in this article, the so called set of τ-integers, denoted by $\mathbb{Z}_\tau$. The set of τ-integers is a quasiperiodic set which stems from Number Theory and Numeration Systems [6]. Loosely speaking, τ-integers are those real numbers whose development in base τ through the greedy algorithm have only positive powers of the base. For example, $1, \tau, \tau^2$ and $\tau^7 + \tau^4 + 1$ are τ-integers, but $2\tau = \tau^2 + 1/\tau$ and $4 = \tau^2 + 1 + 1/\tau^2$ are not τ-integers. The set of τ-integers has the following selfsimilar property

$$\mathbb{Z}_\tau/\tau^{j-1} \subset \mathbb{Z}_\tau/\tau^j \subset \mathbb{Z}_\tau/\tau^{j+1}\,.$$

Starting from this relation, we give a partition of the space of pure point parts of diffractive measures of the sets Λ. The method developed in the following is clearly inspired from multiresolution analysis, such as it is encountered in the theory of Wavelets, see [7]. Indeed, in some previous study we have used the τ-wavelets of Haar [8] to analyze diffraction patterns of one-dimensional aperiodic sets [9]. We shall again use the idea of multiresolution in a future work where we partition the pure point diffractive measure of two dimensional structures [10].

This work was ignited by a study by Gazeau and Krejcar in [11]. They showed that in the diffraction pattern of the Fibonacci chain, Bragg peaks of intensity grater than a certain cutoff $c = (2\sin(\gamma/2)/\gamma)^2$, where $\gamma = 2\pi\tau^2/(\tau^2 + 1)$, are supported by τ-integers.

In the present contribution we give a general method to proceed to a partition of the pure point spectra of any one-dimensional quasicrystal, without restriction on the intensities of the Bragg peaks we consider. Our aim is to give a geometric classification of Bragg peaks arising from diffraction patterns of quasicrystals. Note that we do not tackle the problem of structure determination nor of Patterson analysis directly. We rather propose to discriminate quasicrystalline structures or to compute partial Patterson functions on the basis of this classification.

The organization of this paper is as follows. Section 2 is an overview of the set of τ-integers. Especially we recall the construction of $\mathbb{Z}_\tau$, using the so called Fibonacci substitution, and the Cut and Project schemes associated to $\mathbb{Z}_\tau$ and its inflated or deflated versions. In Section 3 we use the selfsimilarity properties of $\mathbb{Z}_\tau$ in order to partition pure point parts of diffraction patterns of aperiodic sets. In Section 4 we give numerical examples of our partition method on three aperiodic sets. We conclude this article with Section 5, by giving some remarks on the relevance of such partitions in the study of quasicrystals.

2 The set of τ-integers $\mathbb{Z}_\tau$

Recall that the Golden Ratio, denoted by $\tau = (1+\sqrt{5})/2$, is a quadratic Pisot unit, solution of the equation $X^2 = X + 1$. Its Galois conjugate, *i.e.* the other root of this equation, is $\tau' = (1-\sqrt{5})/2 = -1/\tau$.

The Golden Ratio is associated to the *Fibonacci substitution*, denoted by σ_τ, defined on the set of letters $\mathbb{A} = \{L, S\}$ by

$$\sigma_\tau : \begin{cases} L \mapsto LS \\ S \mapsto L. \end{cases}$$

The fixed point of the substitution, denoted by $\sigma_\tau^\infty(L) = LSLLSLSL\ldots$, is associated with an aperiodic tiling of the positive real line, made with the two tiles L and S, where the lengths of the tiles are $\ell(L) = 1$, $\ell(S) = \tau - 1 = 1/\tau$. As such, the nodes of this substitution tiling are the positive τ-integers, $\mathbb{Z}_\tau^+ = \{0, 1, \tau, \tau^2, \tau^2+1, \ldots\}$. It is left to the reader to check that any element of $\mathbb{Z}_\tau^+$ can be uniquely written in the form $\sum_{i=0}^{k} \xi_i \tau^i$ with $k \in \mathbb{N}$ and $\xi_i \xi_{i+1} = 0$, using the identity $\tau^n = \tau^{n-1} + \tau^{n-2}$, $n \in \mathbb{Z}$. The complete set of τ-integers is given by

$$\mathbb{Z}_\tau = \mathbb{Z}_\tau^+ \cup \mathbb{Z}_\tau^- ,$$

where $\mathbb{Z}_\tau^- = -\mathbb{Z}_\tau^+$. One can see that, by construction, the set of τ-integers is symmetric with respect to the origin. The set of τ-integers is a selfsimilar set

$$\mathbb{Z}_\tau/\tau^{j-1} \subset \mathbb{Z}_\tau/\tau^j \subset \mathbb{Z}_\tau/\tau^{j+1} , \qquad (1)$$

where $j \in \mathbb{Z}$.

Another important issue for the rest of the article is the construction of the set of τ-integers, within the *Algebraic Cut and Project* formalism [12]. Denote by $\mathbb{Z}[\tau] = \{m + n\tau \mid m, n \in \mathbb{Z}\}$, the extension ring of the Golden Ratio. We define the *Galois conjugation* as the following automorphism of $\mathbb{Z}[\tau]$

$$' \ : \ x = m + n\tau \quad \mapsto \quad x' = m + n\tau' = m - \frac{n}{\tau} , \qquad (2)$$

where $m, n \in \mathbb{Z}$. Now define the aperiodic set Σ^Ω

$$\Sigma^\Omega = \{x \in \mathbb{Z}[\tau] \mid x' \in \Omega\} ,$$

where Ω, referred to as the *window* of Σ^Ω, is a bounded subset of $\mathbb{R}$ of non-empty interior, see [13] and [14]. The sets of positive and negative τ-integers are given by [16]

$$\mathbb{Z}_\tau^+ = \Sigma^{(-1,\tau)} \cap \mathbb{R}^+ ,$$
$$\mathbb{Z}_\tau^- = \Sigma^{(-\tau,1)} \cap \mathbb{R}^- ,$$

leading to

$$\frac{\mathbb{Z}_\tau^+}{\tau^j} = \Sigma^{(-\tau^j,\tau^{j+1})} \cap \mathbb{R}^+ , \qquad (3)$$

$$\frac{\mathbb{Z}_\tau^-}{\tau^j} = \Sigma^{(-\tau^{j+1},\tau^j)} \cap \mathbb{R}^- \qquad (4)$$

with $j \in \mathbb{Z}$.

Note that this construction is equivalent to the traditional higher dimensional Cut and Project scheme, with the sets $\mathbb{Z}_\tau/\tau^j$ being embedded in the so called parallel space, and their conjugated sets $(-1)^j \tau^j (\mathbb{Z}_\tau)'$ begin embedded in the so called perpendicular space. However one should be careful when discussing the "window" of $\mathbb{Z}_\tau/\tau^j$, since we need a window for the positive part and another window for the negative part, namely $(-\tau^j, \tau^{j+1})$ and $(-\tau^{j+1}, \tau^j)$, respectively.

The selfsimilarity property of $\mathbb{Z}_\tau$, displayed in equation (1) leads to the following result. Define the set of τ-adic numbers as $\mathcal{T} = \{\mathbb{Z}_\tau/\tau^j \mid j \in \mathbb{Z}\}$. In the case of the Golden Ratio we have $\mathcal{T} \subset \mathbb{Z}[\tau]$. Conversely, it can be proven that for all $x \in \mathbb{Z}[\tau]$ there exist a $j \in \mathbb{Z}$ and a $b \in \mathbb{Z}_\tau$ such that $x = m + n\tau = b/\tau^j$. Therefore we have

$$\mathcal{T} = \mathbb{Z}[\tau] .$$

This means that the extension ring of the Golden Ratio is the set of all real numbers whose τ-expansion is finite, see for instance [15], and we can partition $\mathbb{Z}[\tau]$ using the inflated and deflated version of $\mathbb{Z}_\tau$

$$\cdots \subset \tau^i \mathbb{Z}_\tau \subset \cdots \subset \mathbb{Z}_\tau \subset \cdots \subset \mathbb{Z}_\tau/\tau^j \subset \cdots \subset \mathbb{Z}[\tau] . \qquad (5)$$

Denote now by $D_j = \mathbb{Z}_\tau/\tau^j \setminus \{\mathbb{Z}_\tau/\tau^k, k < j\}$, $j \in \mathbb{Z}$, the complement of $\mathbb{Z}_\tau/\tau^{j-1}$ in $\mathbb{Z}_\tau/\tau^j$

$$\frac{\mathbb{Z}_\tau}{\tau^j} = \frac{\mathbb{Z}_\tau}{\tau^{j-1}} \cup D_j .$$

The set D_j can be written in the Cut and Project scheme as

$$D_j^+ = \left(\Sigma^{(-\tau^j,-\tau^{j-1})} \cap \mathbb{R}^+\right) \cup \left(\Sigma^{(\tau^j,\tau^{j+1})} \cap \mathbb{R}^+\right) , \qquad (6)$$

$$D_j^- = \left(\Sigma^{(-\tau^{j+1},-\tau^j)} \cap \mathbb{R}^-\right) \cup \left(\Sigma^{(\tau^{j-1},\tau^j)} \cap \mathbb{R}^-\right) , \qquad (7)$$

where D_j^+ and D_j^- denote respectively the positive part and the negative part of D_j. Most of the proofs of the results given in this section can be found in [16] and references therein.

3 Scale dependent partitioning of diffraction patterns

Let $\Lambda \subset \mathbb{R}$ be a discrete point set. In the reciprocal space, the function giving the intensity per diffracting site is the following limit [11]

$$I(k) = \lim_{N\to\infty} \left| \frac{1}{N} \sum_{\lambda_n \in \Lambda_N} \exp(ik\lambda_n) \right|^2 ,$$

where Λ_N is a chain of N consecutive elements of Λ and λ_n denotes the n^{th} element of the chain. It has been shown by Hof [5] that if $\Lambda = \Sigma^\Omega$ is a Cut and Project set, the above formula well describes the pure point part of the diffraction measure of the set Σ^Ω, *i.e.* the Fourier transform of its autocorrelation, and $I(k)$ is then the following measure

$$I(k) = \sum_{k\in(\Sigma^\Omega)^*} |c_k|^2 \delta_k ,$$

where δ_k denotes the Dirac delta function at k, and c_k is the Fourier coefficient of Λ for the wavelength k. A Bragg peak is then defined as the weighted Dirac measure $|c_k|^2\delta_k$. The set $(\Sigma^\Omega)^*$ is dense in the reciprocal space.

We shall now give a partition of $I(k)$ using the selfsimilar properties of $\mathbb{Z}_\tau$.

Recall that when the Cut and Project scheme of Σ^Ω is associated with the Golden Ratio, with a certain choice of unit in the physical space, the support of the pure point diffractive measure of Σ^Ω is $(\Sigma^\Omega)^* = 2\pi/(\tau^2+1)\mathbb{Z}[\tau]$. Bragg peaks are located at [17]

$$k = k_{||} = \frac{2\pi}{\tau^2+1}(m+n\tau) = \frac{2\pi}{\tau^2+1}\frac{b}{\tau^j} , \tag{8}$$

in the reciprocal parallel space, and at

$$k' = k_\perp = \frac{2\pi\tau^2}{\tau^2+1}(m-\frac{n}{\tau}) = \frac{2\pi\tau^2}{\tau^2+1}(-1)^j\tau^j b' , \tag{9}$$

in the reciprocal perpendicular space, with $m, n \in \mathbb{Z}$, $b \in \mathbb{Z}_\tau$ and $j \in \mathbb{Z}$. We shall discard the scaling factors from now on, and focus on the labelling of the Bragg peaks.

Definition 1 *We say that a Bragg peak belongs to scale $j \setminus (j-1)$, if it is supported by an element of D_j.*

Denote by Π_0 the space of weighted Dirac measures supported by $\mathbb{Z}_\tau$

$$\Pi_0 = \left\{ \sum_{b\in\mathbb{Z}_\tau} |c_b|^2\delta_b \right\} .$$

In the same fashion, denote by Π_1 the space of weighted Dirac measures supported by $\mathbb{Z}_\tau/\tau$

$$\Pi_1 = \left\{ \sum_{b\in\mathbb{Z}_\tau/\tau} |c_b|^2\delta_b \right\} .$$

We obviously have $\Pi_0 \subset \Pi_1$. Denote then by Δ_1 the complement of Π_0 in Π_1, $\Pi_1 = \Pi_0 \cup \Delta_1$. The set Δ_1 defines the space of weighted Dirac measures belonging to scale $1 \setminus 0$

$$\Delta_1 = \left\{ \sum_{b\in D_1} |c_b|^2\delta_b \right\} .$$

Furthermore, we can decompose each scale of the pure point part of the diffraction pattern of Σ^Ω. Let

$$\Pi_j = \left\{ \sum_{b\in\mathbb{Z}_\tau/\tau^j} |c_b|^2\delta_b \right\} ,$$

and

$$\Delta_j = \left\{ \sum_{b\in D_j} |c_b|^2\delta_b \right\} ,$$

for a $j \in \mathbb{Z}$. Viewed against equation (1), the sequence of subspaces $\{\Pi_j\}_{j\in\mathbb{Z}}$ has the following inclusion property

$$\Pi_{j-1} \subset \Pi_j \subset \Pi_{j+1} .$$

We have

$$\begin{aligned} \Pi_{j+1} &= \Pi_j \cup \Delta_{j+1} , \\ &= \Pi_0 \cup \Delta_1 \cup \cdots \cup \Delta_{j+1} . \end{aligned} \tag{10}$$

In the language of wavelets, Π_j is referred to as the space of the *tendency*, and Δ_{j+1} as the space of *details*.

We shall now extend the Galois conjugation map to the weighted pure point measure, for all $x = m + n\tau$, $m, n \in \mathbb{Z}$

$$' : |c_x|^2\delta_x \mapsto |c_x|^2\delta_{x'} .$$

Thus we can define the conjugate space Π'_j of Π_j

$$\Pi'_j = \left\{ \sum_{b'\in(\mathbb{Z}_\tau/\tau^j)'} |c_b|^2\delta_{b'} \right\} ,$$

and the conjugate space Δ'_j of Δ_j

$$\Delta'_j = \left\{ \sum_{b'\in D'_j} |c_b|^2\delta_{b'} \right\} .$$

This definition leads to

$$\Pi'_{j+1} = \Pi'_0 \cup \Delta'_1 \cup \cdots \cup \Delta'_{j+1} . \tag{11}$$

Eventually we have partitioned the pure point part of the diffraction measure, in the normalized reciprocal space, as

$$I(b) = \Pi_0 \cup \left(\bigcup_{j\in\mathbb{N}^*} \Delta_j \right) , \tag{12}$$

$$I(b') = \Pi'_0 \cup \left(\bigcup_{j\in\mathbb{N}^*} \Delta'_j \right) . \tag{13}$$

4 Numerical examples

We would like to illustrate the partitions given by equation (10) and equation (11) with three aperiodic sets generated by the cut and project method.

- $\Sigma^{[0,1)}$, referred to here as the Fibonacci chain,
- $\Sigma^{[-\frac{1}{\tau},0]} \cup \Sigma^{[\frac{1}{\tau^2},1]}$, a two-window cut and project set,
- a set obtained by the concatenation of pieces of cut and project sets, see Appendix for details.

In Figures 1, 2 and 3, on the left hand sides of the figures, we display the pure point parts of the diffractive measures of the above sets, belonging to Π_0, Δ_1, Δ_2, Δ_3 and Δ_4, and the reunion of all the scales in the physical space, for positive $\mathbf{k}$'s. The right hand sides of the figures are the images $(\Pi_0)'$, $(\Delta_1)'$, and so on, in the reciprocal conjugated space, and are slightly more subtle. Recall that the support of Δ_j is given by equations (6, 7). Therefore the support of the conjugated diffraction patterns are

$$\left(\Sigma^{(\tau^{j-1},\tau^j)} \cap \mathbb{R}^-\right)' \cup \left(\Sigma^{(\tau^j,\tau^{j+1})} \cap \mathbb{R}^+\right)',$$

for $j = 1, \cdots, 4$. Hence there is *no* redundancy between the figures from top to bottom, as one could have thought by just looking at the figures.

It is well known that the diffraction pattern of a cut and project set Σ^{Ω} can be computed from the square modulus of the Fourier transform of the set

$$\{x' \in \mathbb{Z}[\tau] \cap \Omega\}.$$

In the case of the Fibonacci chain it has been proven that the diffraction pattern reads

$$I(k) = \left(\frac{\sin(k'/2)}{k'/2}\right)^2,$$

with the notations of equations (8) and (9), see for instance [11]. Once again we discard numerical factors. Therefore in the reciprocal conjugated space, diffraction patterns appear as smooth functions instead of a set of Bragg peaks discretely supported. Thus, although the physical signification of diffraction patterns in the reciprocal conjugated space is not immediate, it appears clearly as a handy tool for investigation. Equation (13) is a decomposition of diffraction patterns in terms of such smooth functions.

The use of such classification of Bragg spectra is to further classify diffractive structures. For example, one may compute diffraction patterns of several aperiodic sets and decompose them as in equations (12) and (13). This work would give a preliminary classification of well known diffractive structures. The idea is then to decompose diffraction patterns obtained experimentally and to compare them with the files that have already been computed. The figures we present show that diffractive structures are well discriminated on the basis of the decomposition we performed, and this discrimination is even clearer in terms of the spaces $(\Delta_j)'$, on the right hand side of the figures. One could also compute partial Patterson function, arising from $(\Delta_j)'$, for some j.

By combining both, classification of diffractive structures and partial Patterson analysis, one may determine diffractive structures more easily.

5 Conclusion

We would like to conclude with three remarks.

Remark 1 Note that when $\Lambda = \Sigma^{\Omega}$ is an aperiodic set, such that its window Ω is a Riemann-integrable convex set, symmetric around the origin, Meyer gave the Fourier transform of the weighted Dirac comb $\mu = \sum_{\lambda\in\Lambda} w(\lambda)\delta_\lambda$ as

$$\hat{\mu} = \sum_{p\geq 1} \nu_p,$$

where ν_p is the weighted Dirac measure supported by the set $\tilde{\Lambda}_p = \Sigma^{p\tilde{\Omega}}$, $p \in \mathbb{N}^*$. In this case, the dual quasicrystal $\tilde{\Lambda}$ is equivalently defined by

$$\tilde{\Lambda} = \{y \in \mathbb{R}, |\exp(iy\cdot\lambda) - 1| \leq 1, \lambda \in \Lambda\},$$

or by the dual convex set $\tilde{\Omega}$. Meyer insisted that the rapid decay of the $\hat{\mu}$ series explained why "*only finitely many layers $\tilde{\Lambda}_p$ are observed in the diffraction pattern of a quasicrystal.*" Scale partitioning of the diffraction pattern is clearly underlying the sums over $\tilde{\Lambda}_p$, see for example [18].

This geometrical partition of the reciprocal space allows to quantify the decay of intensities on the basis of the partition of $\mathbb{Z}[\tau]$ using the set of τ-integers, as in equation (5).

Remark 2 One could argue that such a multiresolution partition of pure point parts of diffraction measures could be performed by using any cut and project set, say $\Sigma^{\tilde{\Omega}}$, provided $\Sigma^{\tilde{\Omega}}$ admits some deflation rule, leading to a "Russian doll" like selfsimilar sequence of cut and project sets embedded in $\mathbb{Z}[\tau]$. And one would be right. This multiresolution partition based on τ-integers is a guideline to help determining which is the proper Russian doll sequence to choose to analyze the pattern. This leads us to the final remark.

Remark 3 At a certain level, this decomposition is not fully satisfactory since we have managed to dissociate the scales, but not the intensities. This work is, to a large extend, a preparation for some future work, where we tackle two-dimensional structures, and where we shall present a modified version of the multiresolution partition of the pure point diffractive measures, combining geometrical partition as show here and intensities partition [10].

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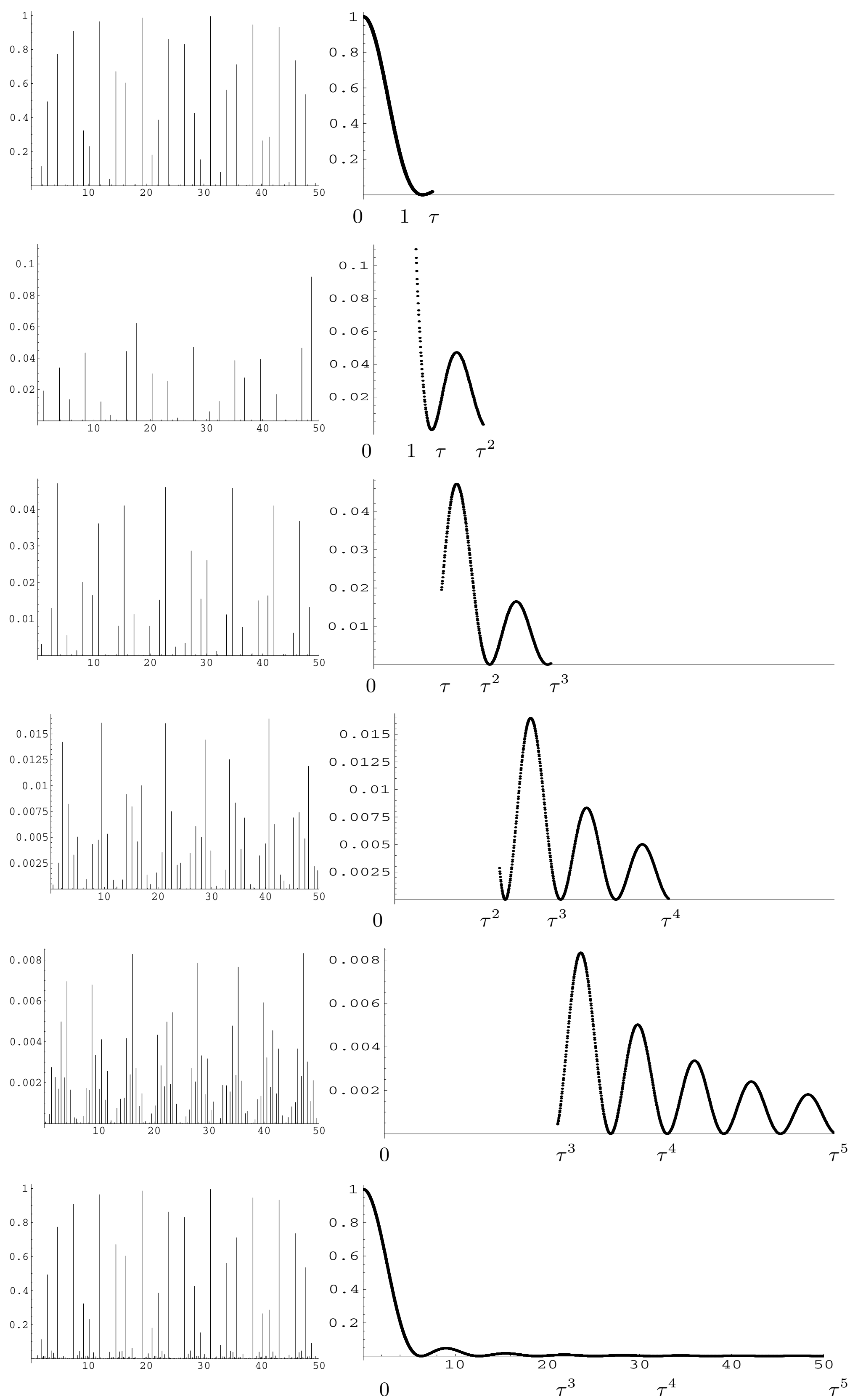

Fig. 1. Multiresolution partition of the diffraction pattern of the Fibonacci chain. Negative parts are obtained by symmetry with respect to the vertical axis. On the left column from top to bottom we display Π_0, Δ_1, Δ_2, Δ_3, Δ_4; on the right column, $(\Pi_0)'$, $(\Delta_1)'$, $(\Delta_2)'$, $(\Delta_3)'$, $(\Delta_4)'$; eventually on the last line is displayed the reconstruction of the whole pattern.

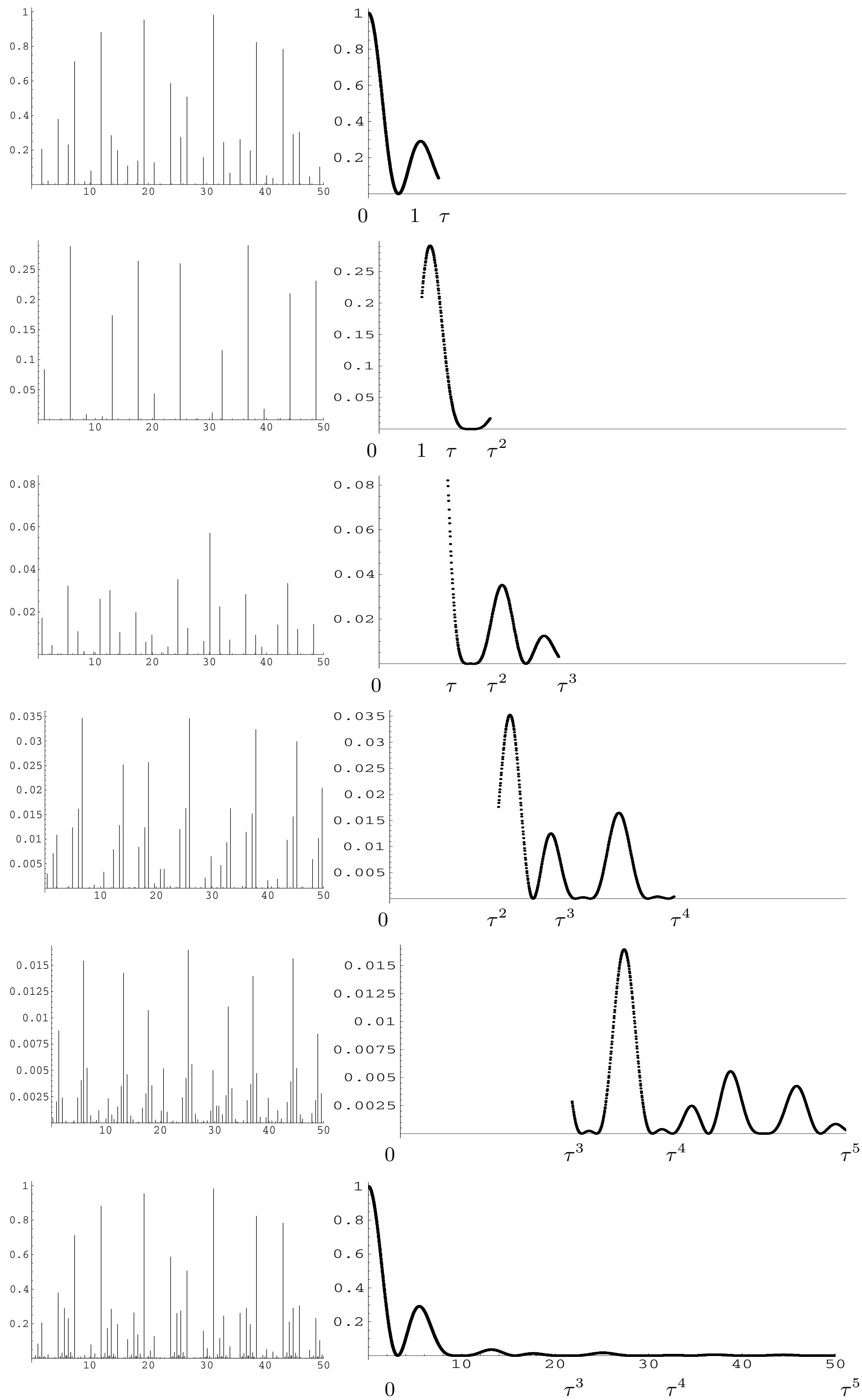

Fig. 2. Multiresolution partition of the pure point part of the diffraction pattern of the aperiodic set $\Sigma^{[-\frac{1}{\tau},0]} \cup \Sigma^{[\frac{1}{\tau^2},1]}$. Negative parts are obtained by symmetry with respect to the vertical axis. On the left column from top to bottom we display Π_0, Δ_1, Δ_2, Δ_3, Δ_4; on the right column, $(\Pi_0)'$, $(\Delta_1)'$, $(\Delta_2)'$, $(\Delta_3)'$, $(\Delta_4)'$; eventually on the last line is displayed the reconstruction of the whole pattern.

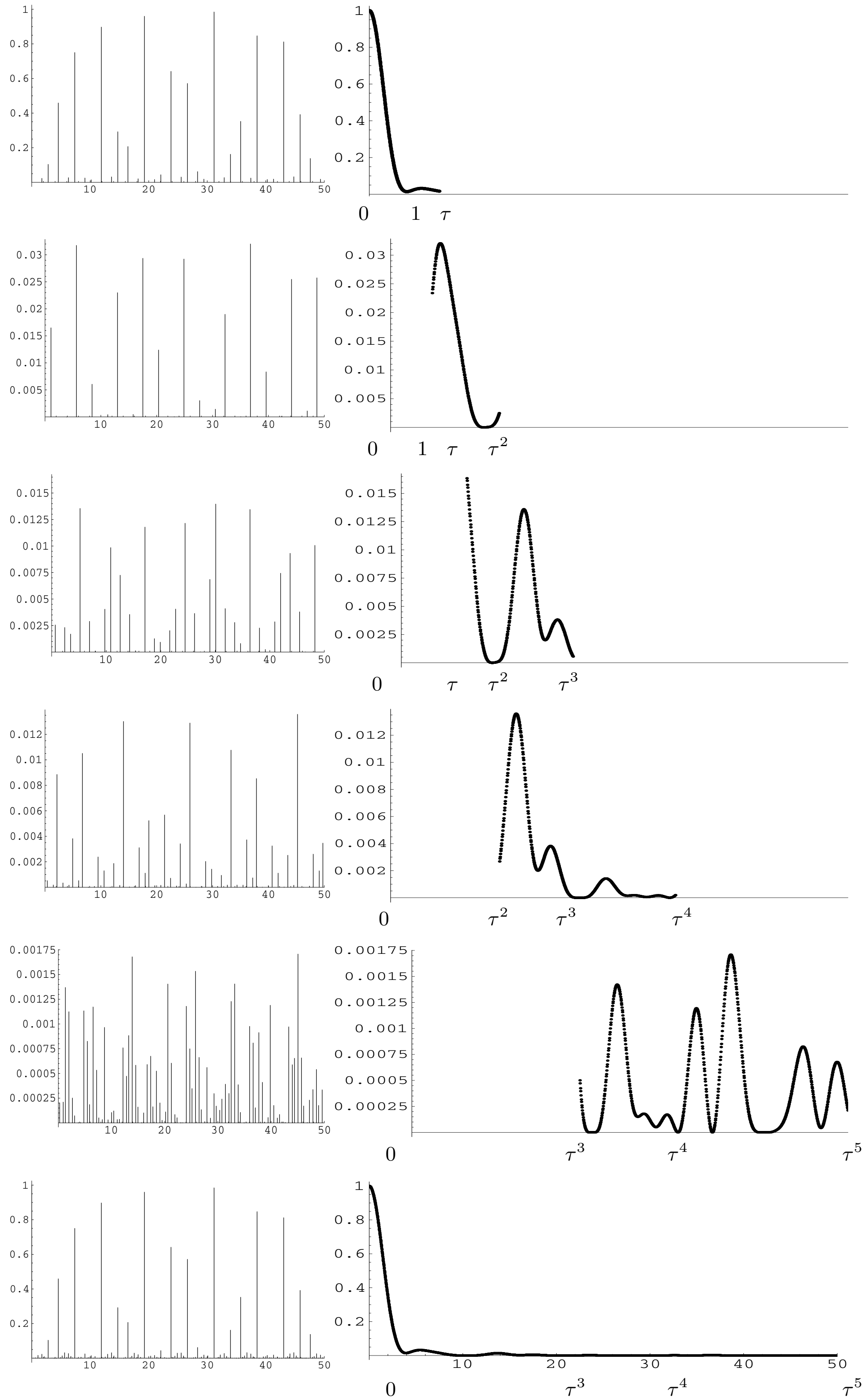

Fig. 3. Multiresolution partition of the pure point part of an aperiodic obtained by concatenation of pieces of cut and project sets. Negative parts are obtained by symmetry with respect to the vertical axis. On the left column from top to bottom we display Π_0, Δ_1, Δ_2, Δ_3, Δ_4; on the right column, $(\Pi_0)'$, $(\Delta_1)'$, $(\Delta_2)'$, $(\Delta_3)'$, $(\Delta_4)'$; eventually on the last line is displayed the reconstruction of the whole pattern.

Appendix

The third model of quasicrystal of which we compute the diffraction pattern is given by the following concatenation of pieces of cut and project sets

$$\begin{aligned} & \Sigma^{(-1/\tau,0)} \cap [-\tau^{20}, \gamma_1[\\ \bigcup\, & \Sigma^{(-1/\tau^2,1/\tau^3)} \cap [\gamma_1, \gamma_2[\\ \bigcup\, & \Sigma^{(1/\tau,1)} \cap [\gamma_2, \gamma_3[\\ \bigcup\, & \Sigma^{(-1,-1/\tau)} \cap [\gamma_3, \gamma_4[\\ \bigcup\, & \Sigma^{(-1/\tau^3,1/\tau^2)} \cap [\gamma_4, \gamma_5[\\ \bigcup\, & \Sigma^{(0,1/\tau)} \cap [\gamma_5, \gamma_6[\\ \bigcup\, & \Sigma^{(-1/\tau^4,1/\tau-1/\tau^4)} \cap [\gamma_6, \tau^{20}[\,, \end{aligned}$$

with

$$\begin{aligned} \gamma_1 &= -(\tau^{19} + \tau^{16} + \tau^8 + \tau^4 + 1)\,, \\ \gamma_2 &= -(\tau^{19} + \tau^{15} + \tau^9 + \tau^7 + \tau^3 + 1)\,, \\ \gamma_3 &= -(\tau^{18} + \tau^{13} + \tau^{11} + \tau^9 + \tau^7 + \tau^5)\,, \\ \gamma_4 &= \tau^{13} + \tau^8 + \tau^3 + \tau\,, \\ \gamma_5 &= \tau^{17} + \tau^{14} + \tau^{11} + \tau^7 + \tau^3 + 1\,, \\ \gamma_6 &= \tau^{19} + \tau^{13} + \tau^9 + \tau^5\,. \end{aligned}$$

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